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## Density reconstruction from multi-directional quantitative Schlieren imaging using Poisson formulation

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### Abstract

This study introduces a framework for reconstructing steady effective line-of-sight-averaged density fields in compressible flows using multi-directional quantitative Schlieren imaging. The experimental setup utilizes a quantitatively calibrated monochrome weak-lens Schlieren configuration to acquire directional density gradient projections across four unique knife-edge orientations ( $0^\circ$ ,  $41^\circ$ ,  $93^\circ$ , and  $-27^\circ$ ), achieving a high signal-to-noise ratio without requiring specialized optics. To solve the overdetermined inverse problem, the individual directional measurements are fused via least-squares estimation to recover the complete spatial gradient components ( $p = \partial\rho/\partial x$ ,  $q = \partial\rho/\partial y$ ). A Poisson equation ( $\nabla^2\rho = f$ ) is then formulated from the divergence of the estimated gradient and solved using the finite element method. The solution is constrained by mixed boundary conditions: homogeneous Neumann conditions upstream, strong Dirichlet conditions downstream, and sparse reference density values enforced as weak Dirichlet anchor points derived from static pressure measurements on the surfaces of the test geometry. A tunable regularization parameter ( $\lambda$ ) governs the enforcement strength of these anchor points, providing structural resilience against high noise levels and data incompleteness. Synthetic benchmarking on a known scalar field validated the algorithm's robustness, demonstrating that high reconstruction accuracy is maintained even under severe noise contamination and sparse anchor distributions, with the optimal regularization parameter shifting toward smaller values as noise levels increase. The framework was applied to a transonic turbine cascade, where the reconstructed field recovered the principal aerodynamic features of the passage flow. Scalar-field accuracy was evaluated separately using synthetic benchmarks with known ground truth. The proposed methodology offers a scalable, versatile tool for quantitative density field visualization across a broad spectrum of flows containing significant pressure or temperature gradients.

## 1. Introduction

The ability to reconstruct accurate density fields by optical diagnostics is crucial for advancing our understanding of high-speed, compressible flows [1, 2]. It enables direct comparison between experimental observations, provides validation for numerical models, and allows for the quantitative analysis of complex flow phenomena such as shock waves, expansion fans, and boundary layer interactions.

Along these lines, Schlieren imaging provides a powerful optical means of visualizing and quantifying refractive index gradients in transparent media. The technique relies on the deflection of light rays as they pass through regions of varying density, governed by Snell's law and linearly related to fluid density through the Gladstone-Dale relation [3, 4]. Classical qualitative Schlieren methods, such as the Toepler knife-edge system, translate these minute refractive index variations into visible contrast, enabling clear visualization of flow features. In contrast, quantitative Schlieren approaches incorporate calibration procedures, which often involve known optical geometries or reference lenses, to convert measured light

deflections into spatially resolved density or refractive index gradients. Advancements such as color (rainbow) Schlieren extended its capabilities by encoding gradient magnitude by color filters, enhancing interpretability. More recently, background-oriented schlieren (BOS) has emerged as a flexible, digital alternative that infers light deflections from apparent displacements in a patterned background, allowing large-field, high-speed, and even outdoor measurements [5].

Building on these developments, recent attention has turned toward quantitative implementations of Schlieren imaging that enable direct measurement of refractive-index and density gradients. Quantitative Schlieren methods allow the extraction of refractive-index gradients and, by extension, density gradients from optical flow visualizations. Among the available techniques, the weak-lens calibrated Schlieren and BOS are widely used due to their relative simplicity and adaptability. In the weak-lens approach, a long focal-length ‘calibration lens’ is introduced into the field of view to establish a known relationship between image intensity and light deflection angle, enabling the conversion of grayscale variations to refractive-index gradients [5]. This technique benefits from the high sensitivity and spatial resolution of a traditional parallel-light Schlieren setup and requires minimal modification to existing optical systems. Conversely, BOS employs a divergent-light configuration in which density gradients are inferred from apparent displacements of a random background pattern, determined through digital image correlation. While BOS provides quantitative Schlieren-like results using simple hardware, a camera and a background, it typically sacrifices spatial resolution and real-time capability [5]. Beyond steady density quantification, Schlieren imaging is also widely used to visualize shock structures, shock–boundary-layer interactions, and flow instabilities. In turbomachinery specifically, high-speed Schlieren has been used to resolve the unsteady interaction between a suction-side normal shock and a laminar separation bubble on a transonic turbine profile, including the characteristic low-frequency shock motion [6], and to study shock–vortical-wake interaction in a transonic turbine nozzle cascade [7]. Similar high-speed Schlieren approaches have revealed instability mechanisms in other flows, such as the Kelvin–Helmholtz and baroclinic origin of helical plumes in atmospheric-pressure plasma jets [8]. The present work complements these time-resolved flow-visualization applications by focusing instead on quantitative reconstruction of the mean line-of-sight-averaged density field from calibrated multi-directional Schlieren measurements. Compared to interferometric methods, both techniques offer cost-effective access to quantitative density gradient data, with the weak-lens system excelling in precision and BOS excelling in flexibility and field of view.

While calibrated color schlieren (CCS) encodes two sensitivity directions in a single exposure using a color filter, it suffers from relatively high noise in the density gradient measurement due to camera noise and additional uncertainty from truncation errors related to color image formatting. Additionally, CCS has a smaller dynamic range compared to BOS, with this range reduced to only 40 in some cases due to the high noise level in the measured density gradient [9]. Extending this approach to multiple directions to create a more general-purpose diagnostic tool requires a multi-band ‘rainbow’ filter, which might increase the calibration complexity and splits the available light across several spectral channels, thus further degrading the signal-to-noise ratio (SNR), amplifying chromatic aberration as well as dynamic range limitations [10]. Moreover, the rainbow filter also causes the diffraction blurring, since the cut-off is performed via the filter transparency [11]. On the other hand, BOS is also vulnerable to image quality issues: camera noise, low-contrast backgrounds, and mechanical vibrations can introduce substantial errors [5, 12–14]. Thus, BOS typically offers lower spatial and temporal resolution than classical Schlieren or shadowgraphy, because it relies on image correlation rather than direct optical focusing. However, with high-speed cameras and dense background textures, modern BOS can achieve good temporal resolution, though still not on par with lens-based Schlieren setups [3, 12, 14, 15]. These quantitative gradients form the foundation for reconstructing full density fields and thermodynamic quantities within complex, compressible flows.

Quantitative reconstruction of density fields from Schlieren images remains a challenging yet essential task in experimental fluid dynamics. Schlieren imaging, provides a visualization of flow features driven by refractive index gradients, which in turn are proportional to density gradients [3]. However, extracting the actual density field from these directional gradient measurements is non-trivial due to the inherent ill-posedness of the inverse problem and the limited information content of Schlieren images, which only capture gradients along specific optical directions. More recently, Bayesian non-parametric models using Gaussian processes introduced calibration-free density inference from knife-edge Schlieren images, validated on supersonic flows in wind tunnels and shock tubes, offering robust uncertainty quantification [16]. Subsequently, phase-consistent dynamic mode decomposition (DMD) reconstructed fluctuating density fields in low-density flows, relevant to turbomachinery under extreme conditions, such as Mars exploration aircraft [17]. Recent advances in data-driven approaches, such as

physics-informed neural networks (PINNs), have improved single-shot Schlieren reconstruction accuracy in asymmetric flows by embedding aerodynamic features, achieving multi-camera-level precision [18]. Quantitative Schlieren combined with Abel inversion has also been applied to map density fields around conical projectiles and overexpanded jets, showing strong agreement with analytical models such as Taylor–Maccoll [19, 20]. In the following, rainbow Schlieren tomography, enabled high-resolution 3D density mapping in underexpanded elliptic jets, resolving shock structures with a spatial resolution of 13  $\mu\text{m}$  using convolution back-projection [21].

Adaptation of some of these scalar field reconstruction methods, has significantly advanced Schlieren density measurements in fluid-flow, combustion, and turbomachinery applications [22]. Rainbow Schlieren deflectometry utilized the Gladstone–Dale relation to quantify refractive index gradients, enabling density field reconstruction in axisymmetric flows with high sensitivity [10]. Subsequent studies introduced CCS and BOS for quantitative density mapping in supersonic flows, validating reconstructed gradients against Prandtl–Meyer theory and PIV measurements [9]. Further developments, such as colored background-oriented schlieren (CBOS), enhanced spatial resolution through multi-color dot patterns and correlation algorithms, facilitating 3D reconstructions in complex supersonic flow fields [23]. An improved CBOS approach further refined density field reconstruction for supersonic flows using algebraic reconstruction techniques, achieving high accuracy in strong shock regions [24]. More recently, calibrated monochrome Schlieren has been applied to highly loaded turbine cascades, using pressure-based boundary conditions for precise density reconstructions critical to analyzing shock losses [22]. While effective, the method relied on a carefully optimized optical and geometric arrangement tailored to the reconstruction task, which constrains its applicability to other turbine geometries and optically challenging experimental setups commonly encountered in aerodynamic research.

Quantitative density-field reconstruction methods based on Schlieren and BOS differ in the source of measurement diversity and in the form of the inverse problem. For nominally two-dimensional flows, BOS commonly provides the two in-plane components of the projected density-gradient vector, from which the scalar field is recovered through direct integration, a Poisson equation, or uncertainty-weighted least squares [13, 14, 25–27]. CCS and colored BOS similarly provide multi-component optical information through spectral or spatial color encoding [9, 23]. These methods permit quantitative planar reconstruction but require a background pattern, color encoding, or specialized calibration and remain subject to line-of-sight integration.

Tomographic BOS instead generates diversity through multiple viewing directions. Reference [15] reconstructed a three-dimensional density field using filtered back-projection, whereas [28] formulated a direct regularized three-dimensional reconstruction from measured deviation fields. These approaches provide volumetric information but require multi-camera or multi-view optical access and solve a substantially larger inverse problem. More recent model-based methods have used derivative-enhanced Gaussian processes with paired Schlieren orientations [16], phase-consistent dynamic-mode decomposition with multi-directional image sequences [17], learned single-shot mappings [18], and physics-informed assimilation of BOS or Schlieren measurements with governing-flow equations [29, 30].

The inherent noise and data incompleteness of optical measurements place strong demands on the reconstruction algorithms used to recover density fields. Traditional image recovery methods from the 1980s [31] provided foundational frameworks for addressing ill-posed inverse problems through techniques like Bayesian reconstruction, convex projections, and Fourier-based phase retrieval. However, they suffered from several key deficiencies that limit their applicability to modern scalar field reconstruction from directional gradients. These approaches often used discrete matrix or continuous-domain formulations or integral projections (for example, in tomography). However, they might face difficulties when applied to dense, overdetermined gradient data, particularly dealing with large computational grids, noise and inconsistencies, as well as poor boundary handling, leading to error accumulation. Moreover, they typically lacked flexible incorporation of sparse reference points via weak constraints, resulting in rigid regularization that could amplify artifacts or fail to generalize beyond specific modalities like CT or astronomy [32]. These limitations hindered simple and accurate density field recovery from Schlieren data where line-of-sight integration and masked flow regions introduce missing or ambiguous data.

This study presents a novel framework for reconstructing steady density fields by combining multi-directional gradient measurements with local static pressure data, as demonstrated in the framework of a transonic turbine cascade. The experimental setup utilizes a quantitatively calibrated monochrome Schlieren technique in a weak lens configuration, where the calibration is achieved by establishing the sensitivity curve between image intensity and refractive index gradients, while multiple gradient projections are acquired through controlled rotation of the knife edge. This offers flexible, high-SNR density gradient measurements without specialized optics or additional calibration. The specific methodological contribution of this work is a single-view, multi-sensitivity quantitative Schlieren architecture

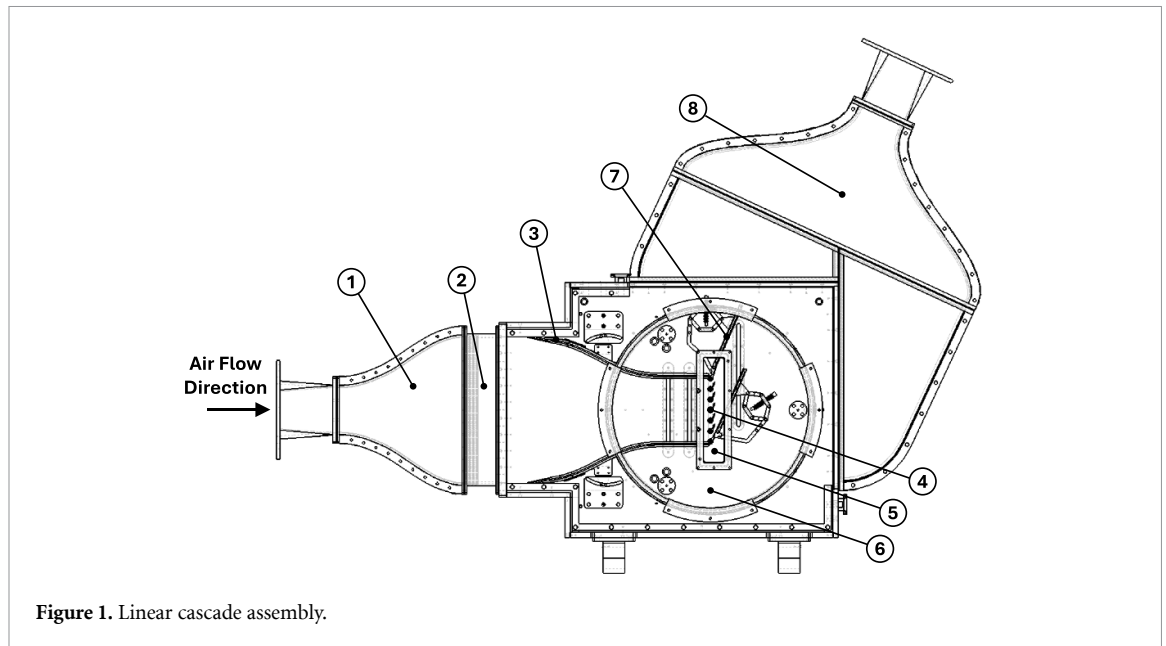


Figure 1. Linear cascade assembly.

in which arbitrary calibrated directional projections are fused through an overdetermined least-squares estimate of the complete in-plane density-gradient vector. The divergence of this dense derivative field defines the source term for a geometry-conforming finite element method (FEM) Poisson reconstruction, while sparse pressure-derived absolute density observations are incorporated through tunable weak constraints together with strong Dirichlet, Neumann, and periodic boundary conditions. This hybrid gradient-anchor formulation accommodates noisy or mutually inconsistent directional measurements and allows the confidence placed in sparse absolute information to be controlled. The methodology is first evaluated against synthetic scalar fields with known ground truth and is subsequently applied to the transonic turbine-cascade measurements.

## 2. Experimental setup

### 2.1. Transonic linear cascade

A transonic linear cascade was used to analyze the sample blades in combination with the Schlieren setup. An overview of the principal subassemblies and the reference coordinate system is provided in figure 1. The section consists of an inlet (1), a flow conditioning module with straightener and turbulence grid (2), adjustable frontboards integrated into the main frame (3), the bladed test section (4), optical access windows (5), rotating disks (6), adjustable perforated tailboards on the main frame (7), and the outlet (8). For the current experiments, the outlet is open to atmospheric conditions. The cascade is driven by a screw compressor in open atmosphere configuration, which can provide a range of  $6 \times 10^4 < Re < 6 \times 10^5$  [33]. The operational  $M_{isen}$  were set by tuning the compressor exit pressure. The whole operation is monitored by a mass flow meter, as well as pressure gauges ( $\sim 0.3\%$  error) between the compressor exit and the cascade inlet. The curve relating the total pressure to the calculated  $M_{isen}$  was extracted experimentally to compensate the piping losses.

Exemplary highly loaded turbine blade configuration was chosen and geometrically scaled to fit within the linear cascade while maintaining solidity, and five blades were mounted via inserts located in the optical windows to preserve flow periodicity [34]. The turbulence intensity was measured at 4.7%, consistent with earlier facility characterizations [33]. The observation windows were fabricated from sapphire, which offers greater than 90% transmission over the 0.15–5.5  $\mu m$  spectral range, and provides strong resistance against abrasion by particles in high-speed flows.

### 2.2. Measurement techniques

#### 2.2.1. Total and static pressure measurements

The cascade was fitted with both total and static pressure measurements to quantify operational conditions, and to impose the blade surface density distributions as anchor points in the reconstruction algorithm. All pressure data were acquired using 16-channel Scanivalve DSA3217 differential transducers, operated in two ranges (0–103 kPa and 0–689 kPa), each with a full-scale accuracy of  $\pm 0.05\%$ .

Static pressures were sampled at 5 Hz over a duration of 60 s, yielding an average based on 300 individual measurements. Upstream and downstream static pressures were measured using 0.5 mm pressure taps with a  $\sim 0.3\%$  error [33, 35]. Inlet and outlet total pressures were obtained with a Vectoflow Kiel probe (0.9 mm head, 3.6% blockage,  $\pm 0.5^\circ$  accuracy for flow angles  $\leq 60^\circ$ ) mounted on a YZ traverse. Inlet uniformity was evaluated at three  $M_{\text{isen}}$  conditions from 45 points (3 spanwise  $\times$  15 pitchwise), yielding spatial uniformity of 99.7% in total pressure and 99.3% in static pressure. Figure 2(a) portrays representative blade surfaces with passage between them, where the pressure taps on suction and pressure sides are indicated on the relevant surfaces. The airfoil profiles are representative of modern high performance turbine geometries with a turning angle of  $53^\circ$ , which geometrically represents the angle between the trailing edge tangent and the horizontal direction. These 2D profiles were extruded into 3D geometries and manufactured on an EOS M290 printer using 17–4 PH stainless steel. Prior to printing the channels, a sacrificial layer was applied to allow additional machining (figure 2(b)). The static pressure channel diameters were set to 0.6 mm, consistent with design guidelines and manufacturing limits. The as-printed blades exhibited an average surface roughness ( $R_a$ ) of 12.2  $\mu\text{m}$ , measured with a Mahr MarSurf PS 10, which was reduced to  $R_a = 0.2 \mu\text{m}$  after CNC finishing (figure 2(c)). Surface static pressures were then acquired at seven discrete locations on both pressure and suction sides of the blades (figure 2(d)). In order to convert the pressure values into density anchor points, the flow is assumed to be isentropic such that upstream total pressure values can be used as a reference to compute the local isentropic Mach number along the blade surface. Furthermore, with the measure of the upstream total temperature, the static value can be calculated considering adiabatic conditions. And, by ideal gas relations, the combination of local static temperature and static pressure can be used to define local density anchor points. Finally, the number of anchor points along the blade surfaces was increased by fitting a continuous curve to the blade geometry and interpolating the measured surface density data along the contour. The pressure-tap values constitute the independent experimental density information; interpolation supplies a continuous numerical representation of that measured boundary distribution and does not create additional independent measurements.

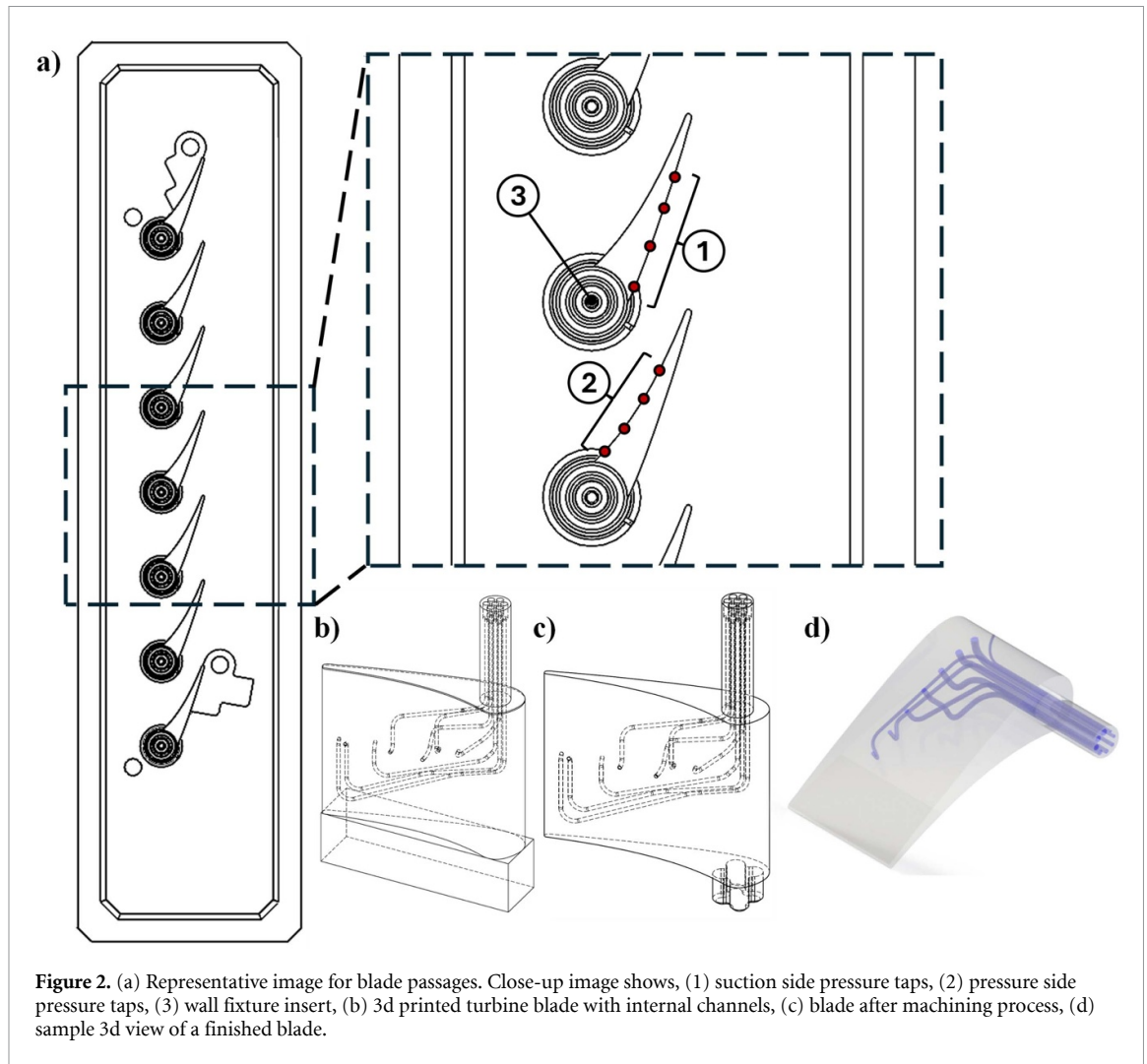
### 2.2.2. Schlieren imaging setup

A Z-type Schlieren arrangement [3, 36] was adopted in the present work, primarily due to the spatial limitations of the optical benches and the need to visualize a large field of view in the cascade test section. The optical configuration used in the experiments is illustrated in figure 3(a).

The system includes the following components: a monochromatic LED source (1), an aspheric condenser diffuser (2), iris-A (3), a condenser lens (4), iris-B (5), plane mirrors (6), parabolic mirrors (7), a knife-edge (8), a high-speed camera (9), optical tables (10), the linear cascade (11), the test blade section (12), and a computer (13). The LED emits light at 528 nm with a luminous flux of 3640 lm. To homogenize the light and minimize scattering, a Thorlabs aspheric condenser diffuser lens ( $f = 40 \text{ mm}$ , 600 grit) was positioned directly after the source. A wide-aperture iris (36 mm maximum opening, iris-A) was then employed to define the beam. The light subsequently passed through a 50 mm condenser lens ( $f/1$ ), followed by a second iris (iris-B, 12 mm aperture), which created a circular and uniform point source before the beam was redirected by a plane mirror. This configuration not only ensured a well-defined beam but also allowed control of the transmitted intensity through the slit-type iris. All upstream elements were mounted on precision posts and aligned along a single optical axis. To block stray light from leaking through the cascade windows and reaching the camera, this initial section of the system was enclosed in a fitted housing.

The collimated light path was achieved using an Edmund Optics plane mirror ( $100 \times 100 \text{ mm}$ ,  $\lambda/4$  protected silver) followed by a parabolic mirror ( $d = 203.2 \text{ mm}$ ,  $f/5$ , 40" FL Enhanced Aluminum). Since iris-B was placed at the focal point of the parabolic mirror, the resulting beam diameter matched the mirror aperture. The collimated beam crossed the test section through sapphire windows. To minimize coma and astigmatism, the parabolic mirrors on either side of the cascade were placed symmetrically about the optical axis, consistent with recommendations in [3]. The relative positioning of all mirrors was carefully adjusted by accounting for focal lengths and minimizing angular offsets. On the downstream side, a mirror arrangement symmetric to the upstream setup refocused the beam between the second plane mirror and the camera. The whole Schlieren system was mounted on two Thorlabs Nexus B6090B optical tables connected by an aluminum breadboard (Thorlabs MB60120/M), supported by four pneumatic isolators (Thorlabs PWA075). This configuration decoupled the system from cascade-induced vibrations and reduced relative motion between the upstream and downstream optical assemblies, ensuring stable knife-edge cutoff during the experiments (figure 3(b)).

Image formation in Schlieren relies on partial light obstruction. In this system, a razor blade was used as the knife-edge, positioned at the focal plane to produce sensitivity to density gradients. Small

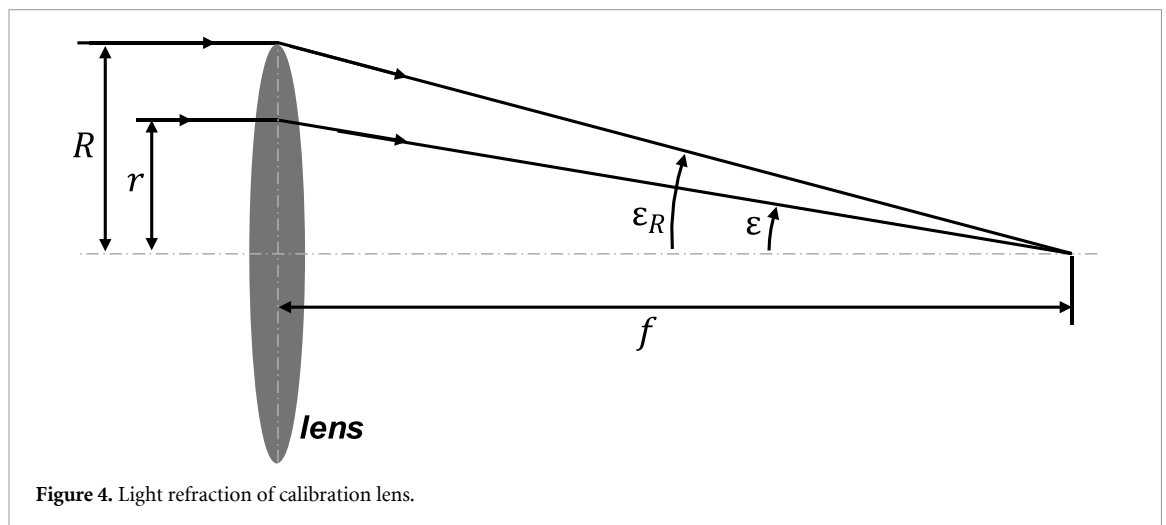
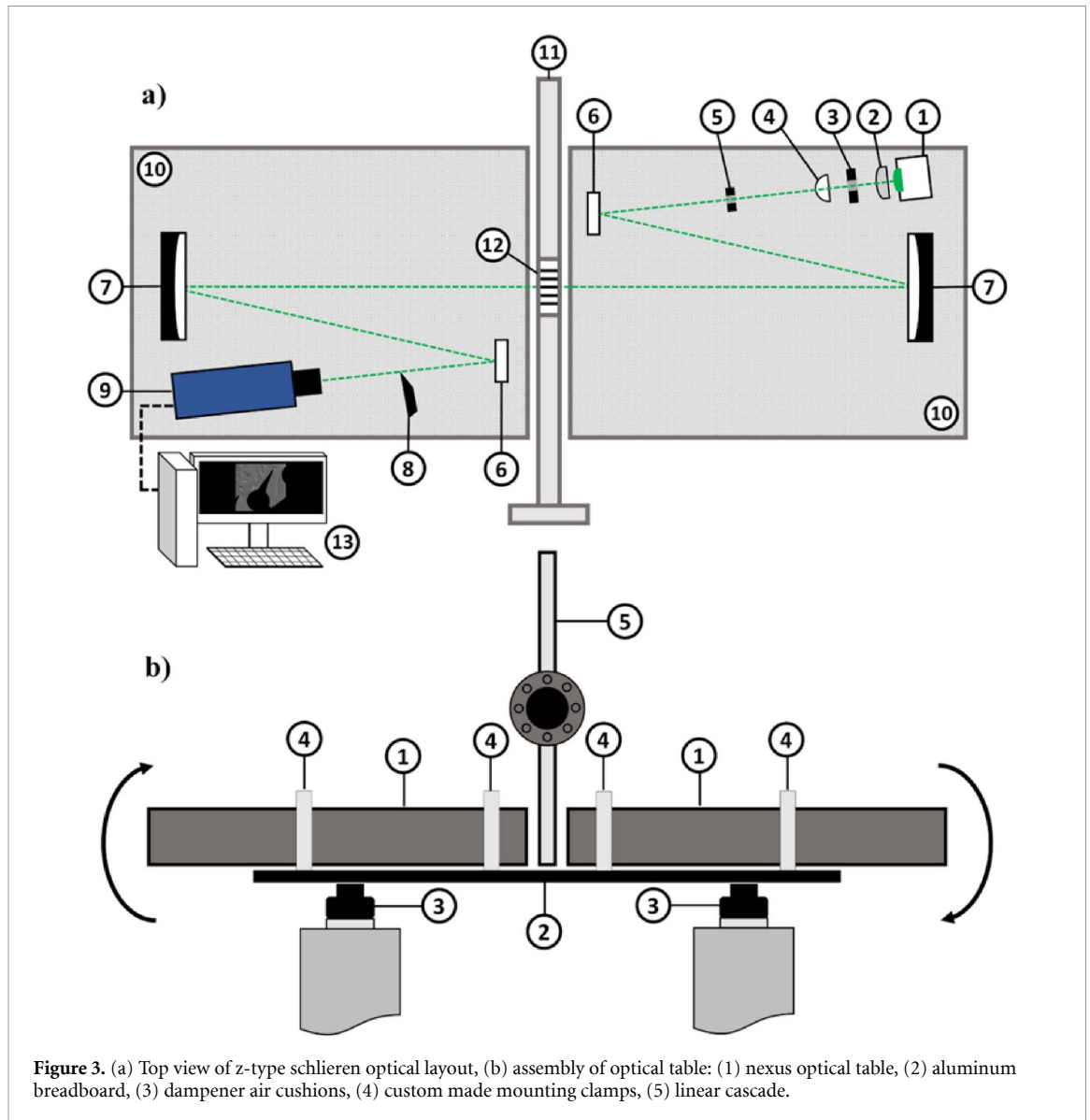


misalignments of the blade resulted in uneven darkening of the background; and correct positioning produced spatially uniform intensity and homogenous dimming of the light intensity with increasing cut-off. Four blade orientations were used in the Schlieren image acquisition, corresponding the cutoff angles of  $0^\circ$ ,  $41^\circ$ ,  $93^\circ$ ,  $-27^\circ$ . The blade angle was defined within the optical cutoff plane, which is perpendicular to the collimated light beam. Angles were referenced with respect to the horizontal direction ( $0^\circ$  corresponding to vertical edge,  $90^\circ$  to horizontal edge). Using IDT Motion Pro Y7-S2 camera with a 150 mm lens operating at 7000 fps, images were acquired for a duration of 14.3 s at each knife position with a resolution of  $512 \times 816$  pixels. Because the imaging data is acquired sequentially, operating-point consistency was checked across the different runs. The inlet total pressure was  $149\,480 \pm 75$  Pa, the corresponding isentropic Mach number was  $0.766 \pm 0.001$ , and the inlet total-temperature spread was 1 K about a mean of 325 K. For the fixed cascade geometry, evaluation of the standard isentropic mass-flux function over these extrema gives an inferred variation of approximately  $\pm 0.2\%$  about the mean.

A calibrated Schlieren technique [3, 5] was employed to quantify the density gradients. Sensitivity is defined as the smallest detectable change in density gradient associated with one resolvable digital-level increment. A dedicated same-orientation repeatability series was not performed; instead, the calibration was independently repeated for each of the four experimental knife-edge configurations at a fixed 50% cutoff with background subtraction. Over the demonstrated range of approximately  $-0.20$  to  $+0.13$   $\text{kg} (\text{m}^3 \cdot \text{mm})^{-1}$ , each calibration is well represented by a linear fit, with an average deviation of approximately 2.4%. Orientation-dependent temporal fluctuations in the measured gradients were approximately 0.3%–2%, and the effective density-gradient sensitivity was approximately  $0.001$   $\text{kg} (\text{m}^3 \cdot \text{mm})^{-1}$ .

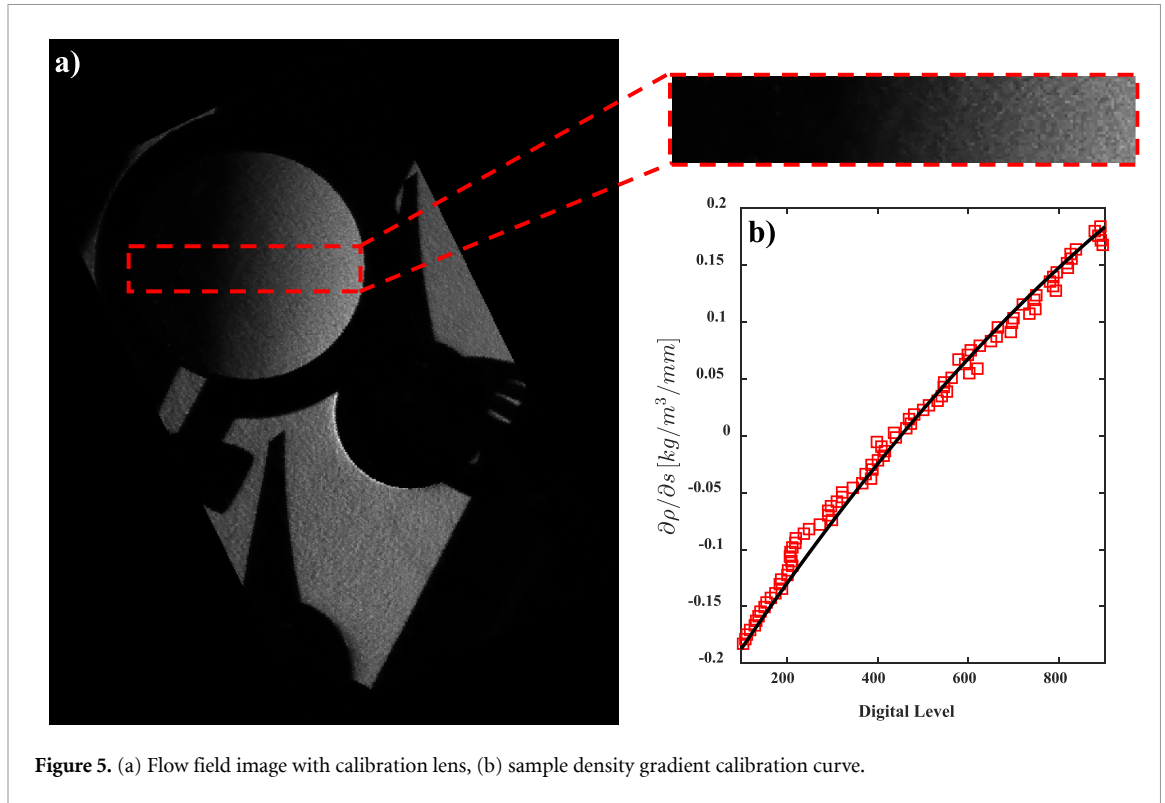
#### 2.2.2.1. Calibration

The calibration procedure begins by quantifying the refraction induced by the weak lens. Following the approach outlined in figure 4 of [5], the lens-induced refraction is evaluated using trigonometric



relations, as expressed in equation (1). Under the assumption of small refraction angles, the tangent function can be approximated by the angle itself, thereby simplifying the calculation.

$$\tan(\epsilon) \approx \epsilon = \frac{r}{f} \quad (1)$$



**Figure 5.** (a) Flow field image with calibration lens, (b) sample density gradient calibration curve.

where  $\epsilon$  is refraction angle,  $r$  is calibration lens radius and  $f$  is calibration lens focal length.

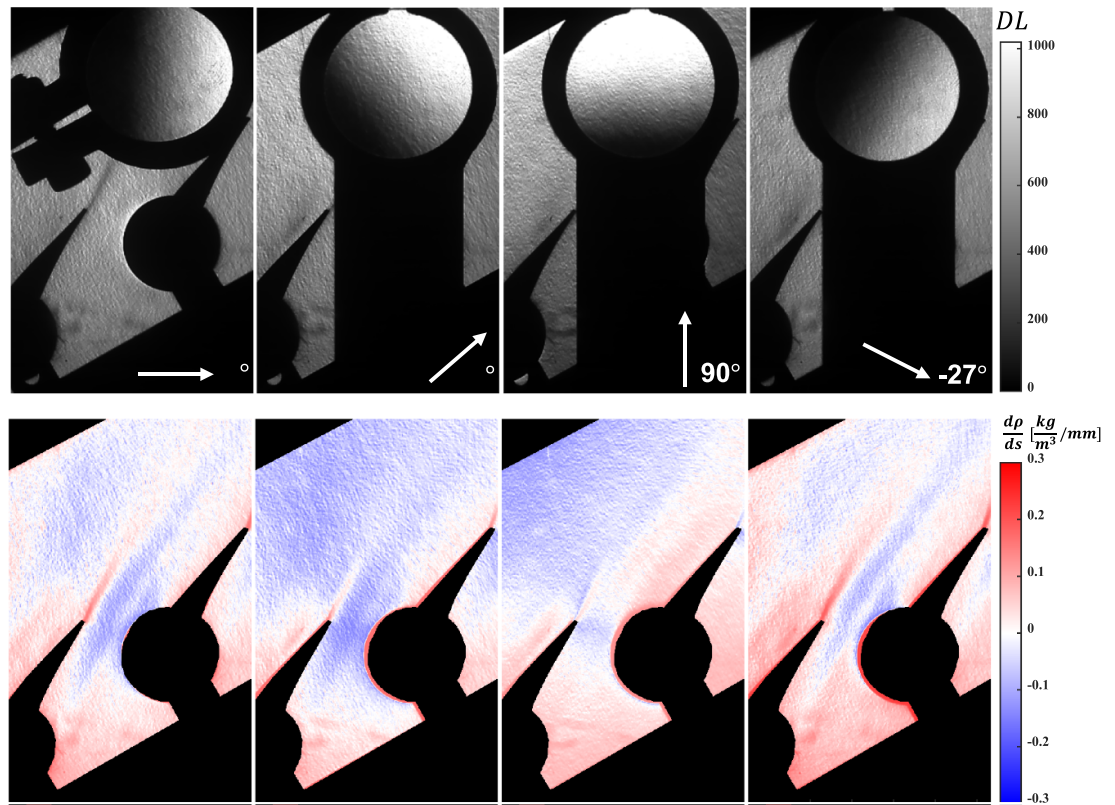
The calibration employs a plano-convex weak lens of diameter  $D = 25.4$  mm ( $+0.00/-0.12$  mm) and nominal focal length  $f = 10$  m ( $\pm 2\%$ ). The corresponding radius is  $R = 12.7$  mm, with a radius tolerance of  $+0.00/-0.06$  mm. Under the small-angle approximation, the maximum one-sided deflection magnitude at the lens edge is  $\epsilon_{\max} \approx 1.27 \times 10^{-3}$  rad. The midpoint of the calibration lens is matched to the uniform-background intensity using a 50% cutoff, and the calibration scale is oriented perpendicular to the knife edge because refraction parallel to the edge produces no measurable intensity change. The resulting grayscale field is mapped to density gradient through:

$$\frac{\partial\rho}{\partial y} = \frac{m_0}{Lkf} \quad (2)$$

Schlieren measures the line integral of the density-gradient component transverse to the optical axis. With the optical path length  $L$  included in equation (2), the calibrated gradient corresponds to the gradient of the path-averaged density,  $\bar{\rho}(x, y) = (1/L) \int_0^L \rho(x, y, z) dz$ . Here  $L = 30$  mm is the spanwise distance between the sapphire windows. The reconstructed  $\bar{\rho}$  therefore has units of  $\text{kg m}^{-3}$  and should be interpreted as an effective line-of-sight-averaged density field. It coincides with a local two-dimensional density only in the limit of spanwise-uniform flow; endwall boundary layers and secondary flows otherwise contribute to the measured average.

A known reference dimension, such as the lens diameter or blade width, can be used to establish the correspondence between the physical scale in meters and the pixel scale of the image. Prior to calibration, the background intensity is subtracted from each calibration image to isolate the optical signal and minimize the influence of ambient illumination. Using equation (2), the density gradient can then be calculated for each pixel as a function of its spatial position. As a result, every pixel is associated with both an intensity value and a corresponding density gradient. By correlating these quantities, the calibration curve is obtained (figure 5).

The lens-geometry uncertainty enters primarily as a scale factor. Because the gradient scale is proportional to  $D/f$ , direct evaluation of the tolerance limits gives a relative scale range of approximately 2.0% about the nominal calibration. If the diameter and focal-length tolerances are treated as independent uncertainty components, their root-sum-square contribution is approximately 2.1%. This value represents the lens-geometry contribution rather than the complete calibration uncertainty. It acts predominantly as a near-uniform multiplicative uncertainty in gradient magnitude and therefore preserves the sign and relative spatial pattern of the measured gradient field.



**Figure 6.** Calibration images (top) and corresponding converted density gradient fields,  $d\rho/ds$  [ $\text{kg}/\text{m}^3/\text{mm}$ ], (bottom) for four different viewing angles.

During conversion of the time-averaged experimental images, the corresponding background image was subtracted to ensure consistent baseline correction. The angles reported for the optical setup denote nominal knife-edge orientations figure 6 (top), whereas the Schlieren sensitivity direction is perpendicular to the knife edge. The true sensitivity direction of each acquisition was recovered independently from the measured grayscale-gradient direction of the radially symmetric calibration lens rather than assumed from the mechanical setting. The deviations between nominal and recovered sensitivity angles were within  $\pm 3^\circ$ , and the recovered directions were used in all subsequent gradient fusion, figure 6 (bottom).

The four knife-edge orientations were not distributed on a regular angular grid but provided independent projections spanning the in-plane density-gradient space. At least two non-collinear orientations are required to recover the two Cartesian gradient components. The additional orientations overdetermine the system, providing measurement redundancy and reducing sensitivity to noise or error in an individual direction. The approximately orthogonal orientations primarily constrain the Cartesian components, while the diagonal orientations provide additional oblique projections of the same gradient field.

#### 2.2.2.2. Image convergence and processing

The reconstruction methodology employed in this study is based on the processing of time-averaged Schlieren images, from which the mean density field is ultimately extracted. To ensure statistical reliability of the averaged quantities, a convergence analysis was conducted to determine the minimum number of images required to achieve stable mean density estimates. The transient Schlieren image sequences were progressively averaged, beginning with 100 frames and extended up to  $10^5$  frames. The full sequence was subdivided into 100 incremental sample sizes. For each case, the standard deviation across all included frames was computed on a pixel-by-pixel basis. These local standard deviation values were subsequently spatially averaged to provide a single representative metric, enabling a quantitative comparison across different sample sizes. The convergence assessment was performed by pairwise comparison of successive sample sizes, where the absolute difference between two cases was used as the metric for evaluation. It is observed that the final population of frames bounds the non-dimensional error for the standard deviation of the means to  $6 \times 10^{-7}$  (see figure 7). At approximately  $10^4$  frames,

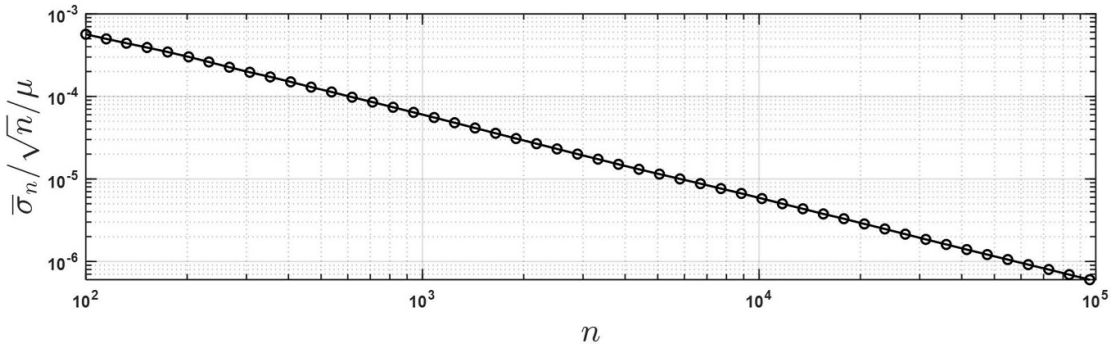


Figure 7. Convergence plot based on the mean standard deviations of various sample sizes.

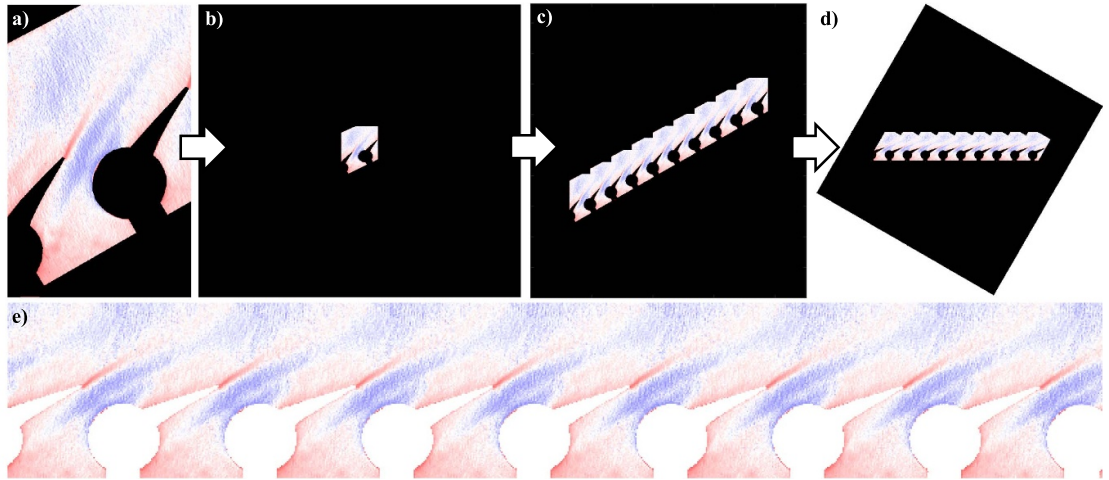


Figure 8. Image manipulation for the reconstruction algorithm. (a) Original gradient field image, (b) padded original image (c) shifted (8 times) and averaged original image with nan field, (d) rotation of the nan-mean image, (e) cropped final image.

the same normalized convergence metric is already reduced to approximately  $6 \times 10^{-6}$ , which is more than two orders of magnitude below the lowest estimated experimental gradient fluctuation of 0.3%. At the acquisition rate of 7000 fps,  $10^4$  frames correspond to approximately 1.4 s of data, indicating that a substantially shorter acquisition already provides a practically converged mean field according to the adopted metric. The  $10^5$  frame sequence was retained as a conservative sampling margin rather than as a minimum requirement of the reconstruction method.

To enhance the periodicity of the flow field representation and suppress nonphysical edge artifacts, an image manipulation procedure was applied to the data. First, the original image (figure 8(a)) was padded with NaN fields to provide sufficient space for subsequent shifts without introducing artificial gradients at the boundaries figure 8(b). The padded image was then shifted eight times, with four shifts in the clockwise direction and four in the counterclockwise direction. The resulting ensemble of shifted images was combined by computing the pixel-wise NaN-mean, which preserves coherent flow structures while attenuating random variations and artifacts (figure 8(c)). Following this averaging process, the image was rotated to align with the desired reference frame and cropped to the computational domain of interest (figure 8(d)). The final processed field (figure 8(e)) provides a periodic and artifact-free dataset from the center passage of cascade test section.

The influence of this processing operation was quantified by comparing a passage from the original measured gradient field with the corresponding registered passage after NaN-padding, shifting, and averaging. A pixel-wise spatial cross-correlation coefficient of  $R = 0.98$  was obtained, demonstrating that the coherent passage-scale gradient structure is preserved while uncorrelated passage-to-passage variations are attenuated. The passage replication is used only to populate the registered central crop during pre-processing.

### 3. Data reduction and reconstruction methodology

In this work, we reconstruct the effective line-of-sight-averaged density field  $\rho(x, y)$  from multiple directional gradient measurements and sparse absolute-density reference values. The approach formulates the problem as a Poisson equation and solves it using a FEM implementation, where both Neumann as well as weak and strong Dirichlet boundary conditions are applied. By combining experimentally measured gradient data with a limited set of anchor points, the method yields a continuous density field. The success criteria are defined by comparing the directional derivative of the reconstructed density field with experimental measurements.

The computational domain is defined as a unit square  $\Omega = [0,1] \times [0,1]$ , and is discretized according to the resolution of the experimental data. After periodicity reconstruction, the grid consists of  $542 \times 1156$  nodes, corresponding to 542 points in the  $x$ -direction and 1156 points in the  $y$ -direction, uniformly distributed over the interval  $[0, 1]$ . Then, the spatial coordinates of the grid points are denoted as  $(x_i, y_j)$ . The available input data comprises four directional gradient fields,  $\{d_1, d_2, d_3, d_4\}$ , each defined over the grid, corresponding to measurements along prescribed orientations of  $[\theta_1, \theta_2, \theta_3, \theta_4]$  as  $[0^\circ, 41^\circ, 93^\circ, -27^\circ]$ , expressed in radians for the computation. Then, the associated unit direction vectors can be compactly represented in a direction matrix:

$$V = \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1) \\ \cos(\theta_2) & \sin(\theta_2) \\ \cos(\theta_3) & \sin(\theta_3) \\ \cos(\theta_4) & \sin(\theta_4) \end{bmatrix} \quad (3)$$

where each direction vector is expressed as  $v_i = [\cos(\theta_i), \sin(\theta_i)]$  and is normalized to unit length  $v_i = \frac{v_i}{\sqrt{\cos^2(\theta_i) + \sin^2(\theta_i)}}$  for  $i = 1, \dots, 4$ . The gradient of the scalar field is denoted as  $\nabla \rho = [p, q]$ , where  $p = \partial \rho / \partial x$  and  $q = \partial \rho / \partial y$ . At each grid point  $(x_i, y_j)$ , the directional derivative along  $(v_k)$  is written as:

$$d_k(x_i, y_j) = \nabla \rho(x_i, y_j) \cdot v_k = p(x_i, y_j) v_{k,x} + q(x_i, y_j) v_{k,y}. \quad (4)$$

With four directional measurements, this forms an overdetermined system:

$$d_{ij} = V \begin{bmatrix} p(x_i, y_j) \\ q(x_i, y_j) \end{bmatrix} \quad (5)$$

where  $d_{ij} = [d_1(x_i, y_j), d_2(x_i, y_j), d_3(x_i, y_j), d_4(x_i, y_j)]^T$ . The least-squares solution for the gradient component is then obtained by:

$$\begin{bmatrix} p(x_i, y_j) \\ q(x_i, y_j) \end{bmatrix} = (V^T V)^{-1} V^T d_{ij}. \quad (6)$$

Providing spatial distributions of  $p(x, y)$  and  $q(x, y)$  across the entire grid. Once the gradients are covered, the density field  $\rho(x, y)$  is reconstructed by solving Poisson equation, where the source term given by the divergence of the estimated gradient:

$$f = \nabla \cdot \nabla \rho = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}. \quad (7)$$

The Poisson problem was discretized using the FEM with a homogenous triangular mesh. The maximum element size was determined through a mesh-dependency analysis, where the element size was varied from 0.2 to 0.0002, covering four orders of magnitude. The optimal size was identified as 0.002, as it marked the onset of a plateau in the correlation coefficient for smaller sizes while remaining computationally efficient. The source term  $f$  is interpolated from the grid values on FEM nodes by linear interpolation.

On the upstream (bottom) boundary  $\partial\Omega$ , homogenous Neumann boundary condition is applied:

$$\frac{\partial \rho}{\partial n} = 0. \quad (8)$$

The side boundaries of the computational domain are prescribed as periodic, while the downstream (top) boundary is defined using a strong Dirichlet boundary condition. Additionally, sparse reference density values, denoted as  $\rho_k$ , are incorporated as weak Dirichlet anchor points  $(x_k, y_k)$  to stabilize the reconstruction and enforce physical consistency. Each anchor is mapped to its nearest mesh node  $n_k$ , and

incorporated through using a formulation with regularization parameter. The stiffness matrix  $A$  and load vector  $b$  are modified as

$$A_{n_k, n_k} = A_{n_k, n_k} + \lambda, \quad b_{n_k} = b_{n_k} + \lambda \rho_k \quad (9)$$

where  $\lambda$  is the regularization parameter controlling the strength of the Dirichlet constraint. The final linear system takes the form:

$$A\rho = b \quad (10)$$

where  $\rho$  represents nodal values of density on the mesh. The system solution is obtained directly as,

$$\rho = A^{-1}b. \quad (11)$$

The nodal solution is mapped back from the FEM mesh to the original grid using linear interpolation, yielding the reconstructed density field  $\rho(x_i, y_j)$ .

To validate the result, the gradients are recomputed on the grid,

$$\nabla\rho = \left[ \frac{\partial\rho}{\partial x}, \frac{\partial\rho}{\partial y} \right] \quad (12)$$

with derivatives evaluated by finite differences. To suppress spurious oscillations, both gradient components are clipped within  $[-1, 1]$ :

$$\frac{\partial\rho}{\partial x} = \min\left(\max\left(\frac{\partial\rho}{\partial x}, -1\right), 1\right), \quad \frac{\partial\rho}{\partial y} = \min\left(\max\left(\frac{\partial\rho}{\partial y}, -1\right), 1\right). \quad (13)$$

The directional gradient along the first direction vector  $v_i$  is evaluated as,

$$d_\rho(x_i, y_j) = \frac{\partial\rho}{\partial x}(x_i, y_j) v_{i,x} + \frac{\partial\rho}{\partial y}(x_i, y_j) v_{i,y} \quad (14)$$

and compared against the corresponding measured values  $d_i(x_i, y_j)$ . The reconstruction accuracy is quantified using the root mean square error (RMSE):

$$RMSE = \sqrt{\frac{1}{N} \sum_{i,j} (d_\rho(x_i, y_j) - d_i(x_i, y_j))^2}, \quad N = 542 \times 1156. \quad (15)$$

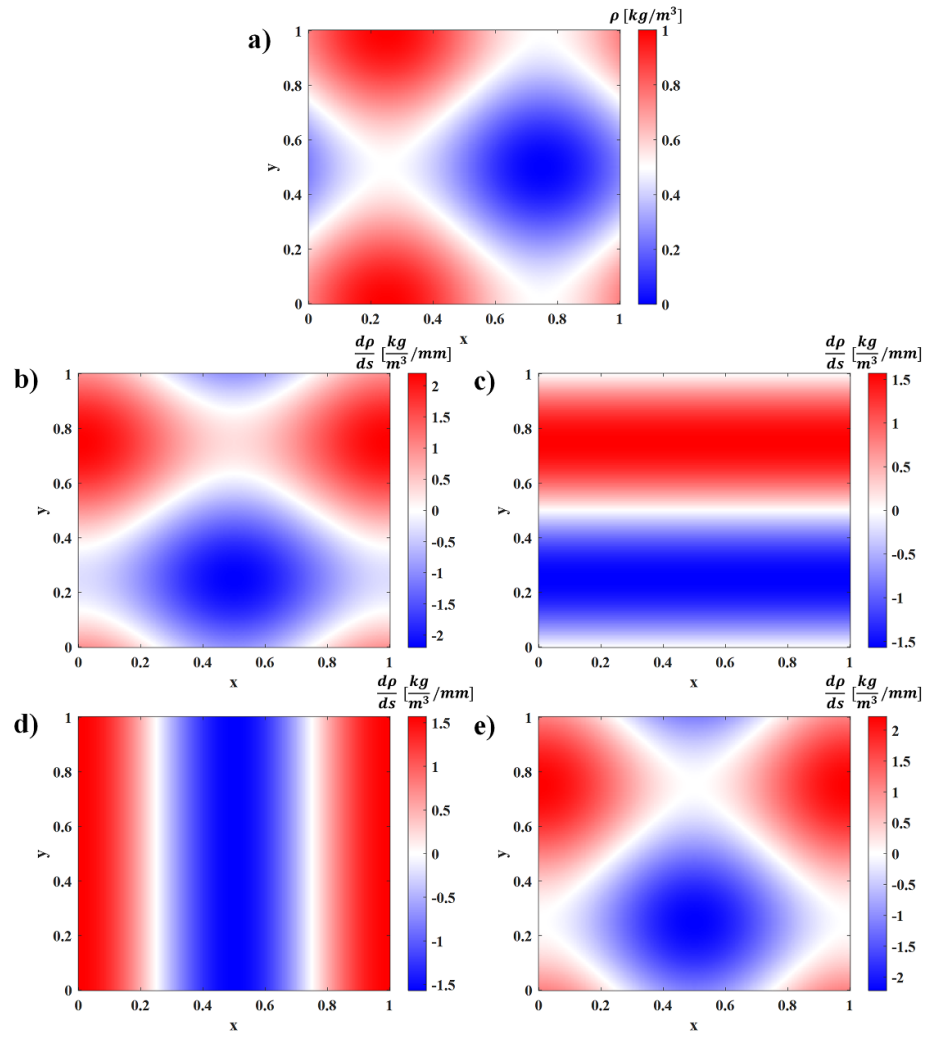
## 4. Synthetic data benchmarking

The proposed framework reconstructs a continuous density field that is consistent with the directional gradients while also anchored by sparse reference values. The FEM-based formulation with weak Dirichlet anchor points provide resilience against noise and incomplete data, ensuring that the final solution preserves the essential flow features and is suitable for further quantitative analysis.

### 4.1. Scalar field and directional gradients

To evaluate the reconstruction algorithm, a synthetic test case was designed with a known ground-truth scalar field and corresponding directional gradients. This controlled setup, incorporating realistic spatial variations and controlled noise to emulate measurement uncertainty, provides a consistent benchmark for quantifying the method's accuracy and robustness before applying it to experimental data. The test field was generated on a  $200 \times 400$  grid over the domain  $[0, 1] \times [0, 1]$  and defined as  $u(x, y) = \sin(2\pi x) + \cos(2\pi y)$ . After normalization to the interval  $[0, 1]$ , directional gradients were computed along four prescribed orientations ( $53^\circ, 90^\circ, 0^\circ, 45^\circ$ ), corresponding to the measurement directions used in the reconstruction framework. The resulting synthetic scalar field (figure 9(a)) together with the four directional gradient fields with the same orientation are illustrated in figure 9.

Initialization was done with a large fraction of grid points set to zero to fill the unknown parts in the domain. The bottom edge of the domain is set as a Neumann boundary condition in solution algorithm. The top edge is defined as Dirichlet constraint. In addition, on the side boundaries, weak sparse



**Figure 9.** (a) Synthetic scalar field, (b) 53°, (c) 90°, (d) 0° and (e) 45° directional gradient fields.

Dirichlet anchor points are introduced at selected locations which were used as the known measurements available for reconstruction, while the remaining side edges are treated with Neumann boundary conditions. To mitigate the influence of strong gradients during reconstruction, the Dirichlet anchor points were relaxed by introducing a uniform regularization factor ( $\lambda$ ) across all anchor points. Without relaxation, strictly enforced Dirichlet conditions can overly constrain the solution in the vicinity of anchor points, leading to sharp discontinuities and reduced physical consistency in the reconstructed field. The application of  $\lambda$  effectively softens the anchor enforcement, providing a balance between adhering to measured boundary data and maintaining smoothness across the interior domain. This setup reflects realistic experimental conditions, where only partial information can be obtained; and the resultant numerical domain is presented in figure 10. Following the algorithm described, the reconstruction method successfully produced the original density field from the gradients, as was defined in the first place (figure 9(a)), thereby validating the approach.

The robustness of the reconstruction algorithm was tested by adding Gaussian noise introduced both to the anchor points and to the measured gradient fields, for a reference regularization parameter  $\lambda = 0.5$ . Although the exact level of noise originating from Schlieren measurements cannot be predetermined, the sensitivity of the reconstruction framework can be systematically assessed by introducing controlled noise into synthetic gradient fields. For the first assessment of the algorithm, the absolute mean value of each gradient field was scaled by a noise multiplier varying between 0 to 1 and added to the original signal. The performance of the method was quantified by RMSE and the cross-correlation coefficient ( $R$ ) between the original and reconstructed density fields. The noise percentage was calculated by the ratio of noise field to signal field deviations. When the mean-bound noise method was applied (figure 11) within the selected range, the RMSE increased from 0.013 to 0.038, while the correlation

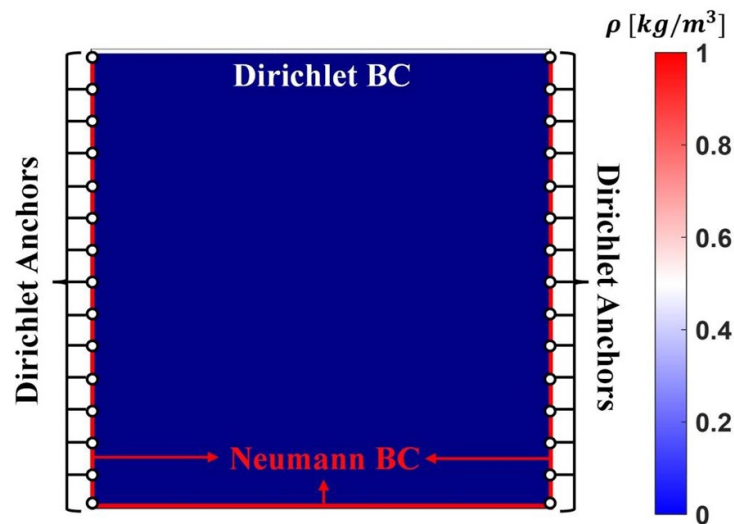


Figure 10. Schematic of the scalar sparse field.

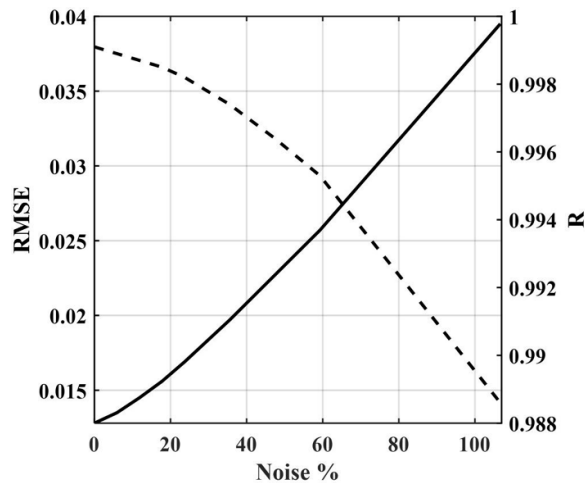
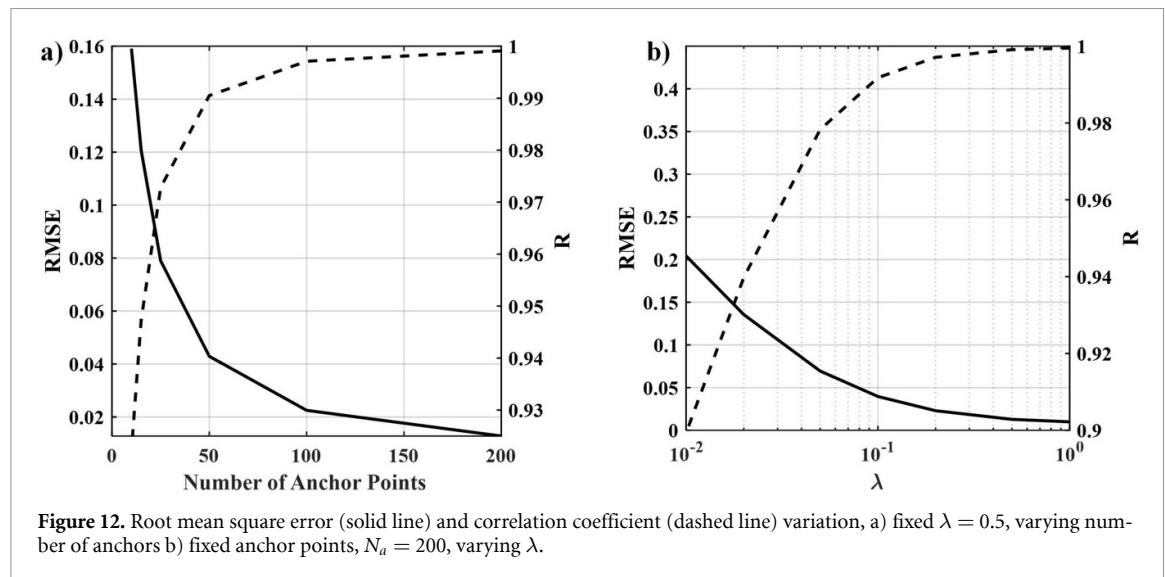


Figure 11. Root mean square error (solid line) and correlation coefficient (dashed line) variation regarding case sensitive absolute mean bounded Gaussian salt & pepper noise.

coefficient  $R$  exhibited a slight degradation of from 0.99 to 0.988. The sensitivity of the reconstruction to a calibration-scale error was also assessed by uniformly rescaling the synthetic input gradients by  $\pm 2.4\%$ , a representative perturbation comparable in magnitude to the measured calibration contributions. This perturbation changed the gradient-field RMSE by approximately 4%, while the reconstructed density-field RMSE changed by only approximately 0.2%. The reduced sensitivity of the scalar field arises because the fixed density anchors constrain the effect of a near-uniform gradient-scale perturbation.

The next evaluation was conducted to monitor the dependency of anchor point sparsity at the left- and right-hand side of the test domain. The significance of this test is to assess the capability of the algorithm to handle increasing sparsity of the Dirichlet boundary conditions. Starting with 200 anchor points (equal to the number of pixels along the  $y$ -axis), the number of points was gradually reduced to 5% of the original at fixed  $\lambda = 0.5$ . The resulting reconstructed field was compared with originally created density field. Figure 12(a) shows that the RMSE increased from 0.013 to 0.16, while the correlation coefficient  $R$  decreased from 0.99 to 0.92 as the number of anchors was reduced from 200 to 10. Within the acceptable correlation range ( $R > 0.95$ ), accurate results were maintained down to 7.5% anchor density (15 points). These findings indicate that, even under extremely sparse initial conditions, the reconstruction algorithm remains capable of delivering reliable results. For consistency in subsequent tests, however, the original choice of 200 anchor points was maintained, as it provided the highest accuracy with  $R = 0.99$ .

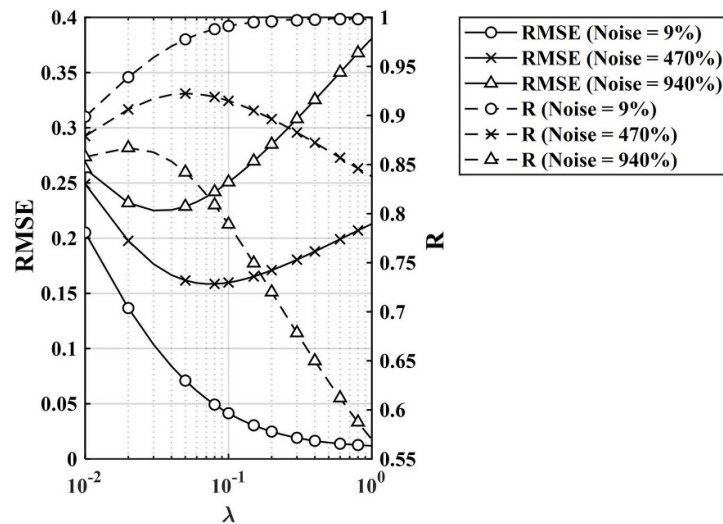


In the next part of the analysis, the influence of the regularization factor  $\lambda$  was examined. The parameter was systematically varied over the range  $10^{-2} \leq \lambda \leq 1$  (see figure 12(b)). Across this interval, the RMSE decreased by approximately 98%, while the correlation coefficient  $R$  increased by 23%, asymptotically approaching 1, highlighting the strong sensitivity of the reconstruction accuracy to the choice of  $\lambda$ . It should be noted that the role of  $\lambda$  cannot be fully characterized in isolation, as its effect is closely coupled with other parameters of the reconstruction framework. A more comprehensive assessment thus requires considering the combined impact of  $\lambda$  and additional factors such as anchor point noise introduction.

To further evaluate the robustness of the reconstruction algorithm, the influence of the regularization factor  $\lambda$  was analyzed under varying anchor noise levels at 100% anchor point density. Figure 13 presents the variation of RMSE and correlation coefficient  $R$  as a function of  $\lambda$  for three representative cases: low noise (9%), moderate noise (470%), and high noise (940%). At low noise (9%), the algorithm exhibits highly uniform behavior across a broad range of  $\lambda$ . The RMSE decreases sharply with increasing  $\lambda$ , reaching a minimum at approximately  $\lambda \approx 0.8$ , beyond which further increases in  $\lambda$  cause only marginal improvements. The correlation coefficient  $R$  remains consistently high (close to 1), confirming that the reconstruction retains excellent agreement with the reference scalar field in the low-noise regime. When the anchor noise level was increased to 470%, the behavior of both error metrics became more pronounced. The RMSE curve exhibited a distinctly non-linear trend and remained consistently higher than in the low-noise case, confirming that the reconstruction error is primarily governed by the added noise, largely independent of  $\lambda$ . A local minimum in RMSE was observed at  $\lambda = 0.08$ , while the correlation coefficient  $R$  reached a local maximum at  $\lambda = 0.05$ . These findings suggest that, under higher levels of anchor noise, smaller values of the regularization parameter yield more favorable performance. Despite the increased RMSE, the correlation coefficient  $R$  decreased only moderately, indicating that although noise amplifies the magnitude of reconstruction error, the global structure and trends of the scalar field are still well captured.

Under extreme noise conditions (940%), the deterioration in reconstruction accuracy becomes particularly evident. The overall RMSE rises sharply and exhibits strongly non-linear behavior, reflecting the dominant influence of noise on reconstruction. A local minimum in RMSE is observed near  $\lambda = 0.03$ , while the correlation coefficient  $R$  attains its local maximum at  $\lambda = 0.02$ . This shift toward smaller optimal values of  $\lambda$  indicates that, as noise levels increase, a weaker anchor enforcement (smaller  $\lambda$ ) becomes more effective in maintaining accuracy. In contrast to the moderate-noise case, the correlation coefficient  $R$  undergoes a pronounced decline alongside the elevated RMSE, signaling that the reconstruction not only accumulates larger errors but also loses its ability to preserve the essential flow structures when higher  $\lambda$  values are applied.

Taken together, these results demonstrate that the algorithm maintains robustness across a range of  $\lambda$  values, even under severe noise contamination; however, the fidelity of the reconstruction depends on the selection of the regularization parameter. For low noise conditions, expectedly, imposing the Dirichlet condition as a rigid anchor point ( $\lambda \rightarrow 1$ ) seems beneficial to the reconstruction of the scalar



**Figure 13.** Root mean square error (solid line) and correlation coefficient (dashed line) variation regarding  $\lambda$  sweep on different anchor noise conditions.

field. However, in moderate noise conditions, an intermediate optimal  $\lambda$  (0.05–0.1) offers the best possible scalar field recovery. Furthermore, it is observed that increasing the noise levels further shifts the optimal  $\lambda$  towards smaller values. This underscores the importance of tuning  $\lambda$  relative to the noise environment to achieve the best reconstruction accuracy. This behavior has a statistical and physical interpretation in terms of the trust placed in the anchor data. The parameter  $\lambda$  sets the enforcement strength of the weak Dirichlet anchors: as  $\lambda \rightarrow \infty$  the anchors approach hard constraints, whereas as  $\lambda \rightarrow 0$  the solution is governed primarily by the divergence-of-gradient source term. Accordingly,  $\lambda$  controls the relative weighting between fidelity to the anchor values and consistency with the density field implied by the interior gradient measurements, representing a bias–variance trade-off [37]. When the anchors are clean, stronger enforcement is beneficial because they accurately pin the scalar level. As anchor noise increases, strong enforcement injects the corrupted values into the reconstruction; reducing  $\lambda$  down-weights the unreliable anchors and gives greater relative weight to the overdetermined, noise-averaged gradient field. Consistent with this interpretation, the preferred  $\lambda$  decreases from approximately 0.8 at 9% anchor noise to 0.05–0.08 at 470% noise and 0.02–0.03 at 940% noise.

To test the reconstruction beyond the smooth periodic benchmark, an additional synthetic scalar field was constructed containing an oblique discontinuity, a localized Gaussian feature, and a non-periodic linear background. The prescribed scalar field and its reconstruction were evaluated using the same [0,1] normalization. The directional gradients were generated from the prescribed field and processed using the same gradient-recovery and Poisson-reconstruction procedure, with boundary conditions consistent with the non-periodic field. The reconstruction recovered the location and overall jump amplitude of the discontinuity together with the localized and non-periodic components. The sharp jump was smoothed over several reconstruction elements, as expected from the finite FEM resolution, while the normalized scalar-field RMSE was 5% of the prescribed full-scale range. Figure A1 in appendix shows the comparison between original and the reconstructed synthetic fields with oblique discontinuity.

## 5. Experimental results

In an experimental setting, in order to apply this reconstruction method, it is essential to establish a framework that provides sufficiently rich data (multiple directional gradient measurements) and consistent well-defined boundary conditions. Weak Dirichlet condition was applied at the anchor locations, while Neumann boundary conditions were imposed at the upstream boundaries of the flow field. The downstream boundary was defined as a Dirichlet condition, corresponding to atmospheric conditions. Periodic boundary conditions were enforced following twelvefold domain replication and subsequent

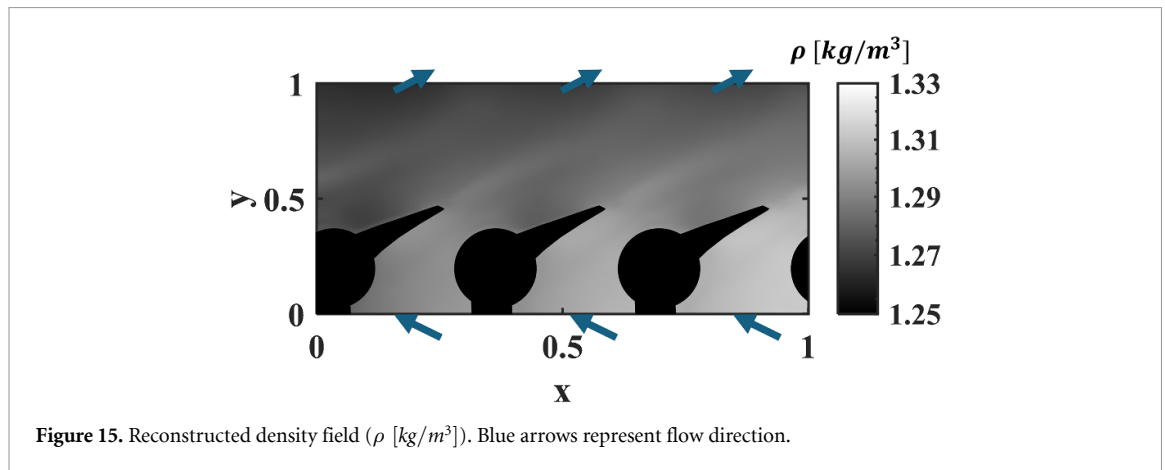
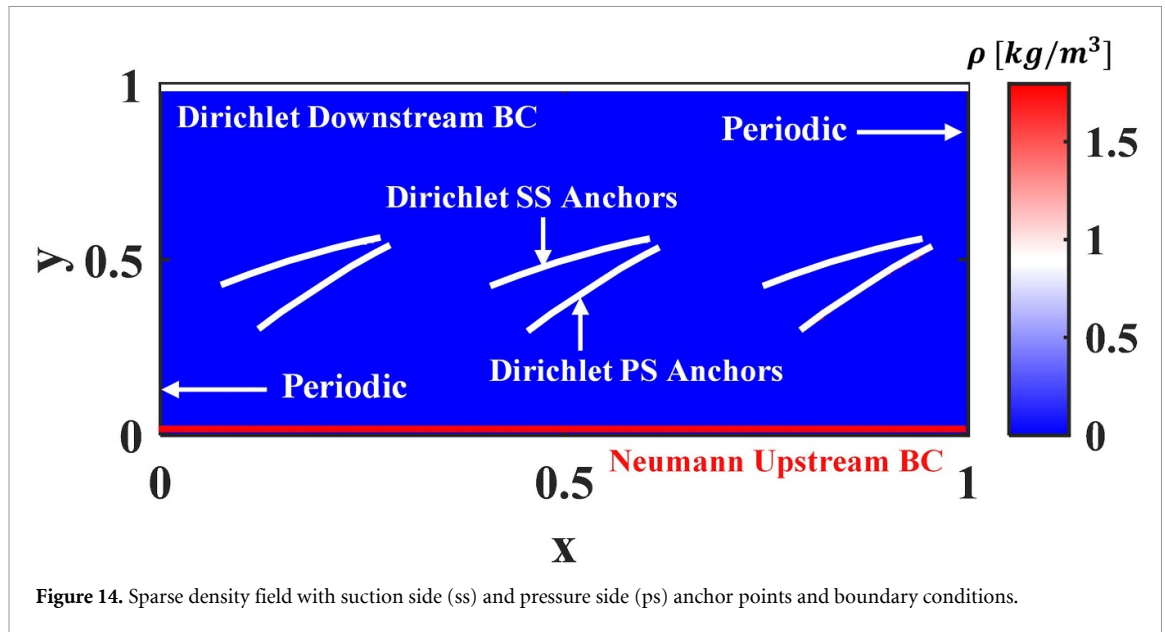


image processing (see figure 8). The remaining region of the field was initialized to zero. After rotation and cropping, the initial density field is presented in figure 14.

The homogeneous upstream Neumann condition is supported by the inlet survey, which yielded 99.5% transverse pressure uniformity over the measurement plane. The lateral periodic condition follows from the geometric periodicity of the cascade. The downstream density was prescribed from the outlet condition and checked against an independent static-pressure measurement 25 mm downstream of the reconstruction region. That measurement gives 1.224 kg m<sup>-3</sup>, compared with approximately 1.25 kg m<sup>-3</sup> imposed at the boundary, a difference of about 2.1%. This comparison supports the order of magnitude of the prescribed outlet density and provides an experimental uncertainty scale for the absolute level near the outlet.

With the gradient information along different directions extracted from four distinct knife edge angles, the reconstruction was carried out using the algorithm outlined in section 3. As concluded in the synthetic data benchmarking section, the Dirichlet anchor points were relaxed using a uniform regularization factor ( $\lambda$ ), logarithmically varied between 10<sup>-3</sup> and 1. In the following, the gradients of the reconstructed field were extracted along the four original directions. The recovered density field obtained under these conditions is shown in figure 15.

The density field illustrates the compressible flow evolution through the blade passage. The flow enters from the bottom at elevated density and undergoes continuous acceleration along the blade passage due to the reduction in effective area until the throat region, leading to a gradual decrease in density. In particular around the suction side, the flow initially undergoes a rapid density reduction due to

flow turning (the view is partially obstructed by the blade inserts), associated with flow acceleration, followed pressure recovery in the aft section manifested as a local density rise. Near the trailing edge, a low-density region is observed in close proximity, consistent with the finite-thickness base region and the Kutta condition [38]. The high-density bands observed downstream of each trailing edge correspond to elevated density because this is a near stagnation region where static pressure has recovered, while the surrounding flow, having accelerated through the passage, exists at significantly reduced static pressure and density. Although total pressure loss and viscous heating within the wake both act to reduce local density, these effects are outweighed by the pressure recovery associated with flow deceleration. Farther downstream, the density progressively recovers toward the ambient value as the wake mixes out and the flow decelerates.

Furthermore, the calculated density distribution is used to regenerate the gradient fields which are directly compared against the experimentally measured Schlieren gradients. Figure 16 presents this comparison for each knife-edge sensitivity angle within the turbine cascade, where the left column shows the experimentally measured gradient fields and the right column presents the corresponding numerically computed gradients. The blue arrows indicate the flow direction, while the red arrows denote the sensitivity direction of each Schlieren orientation.

Each sensitivity direction emphasizes a different projection of the in-plane density-gradient vector. The approximately orthogonal directions primarily constrain the two Cartesian components, whereas the diagonal directions contain contributions from both components and provide redundant directional information.

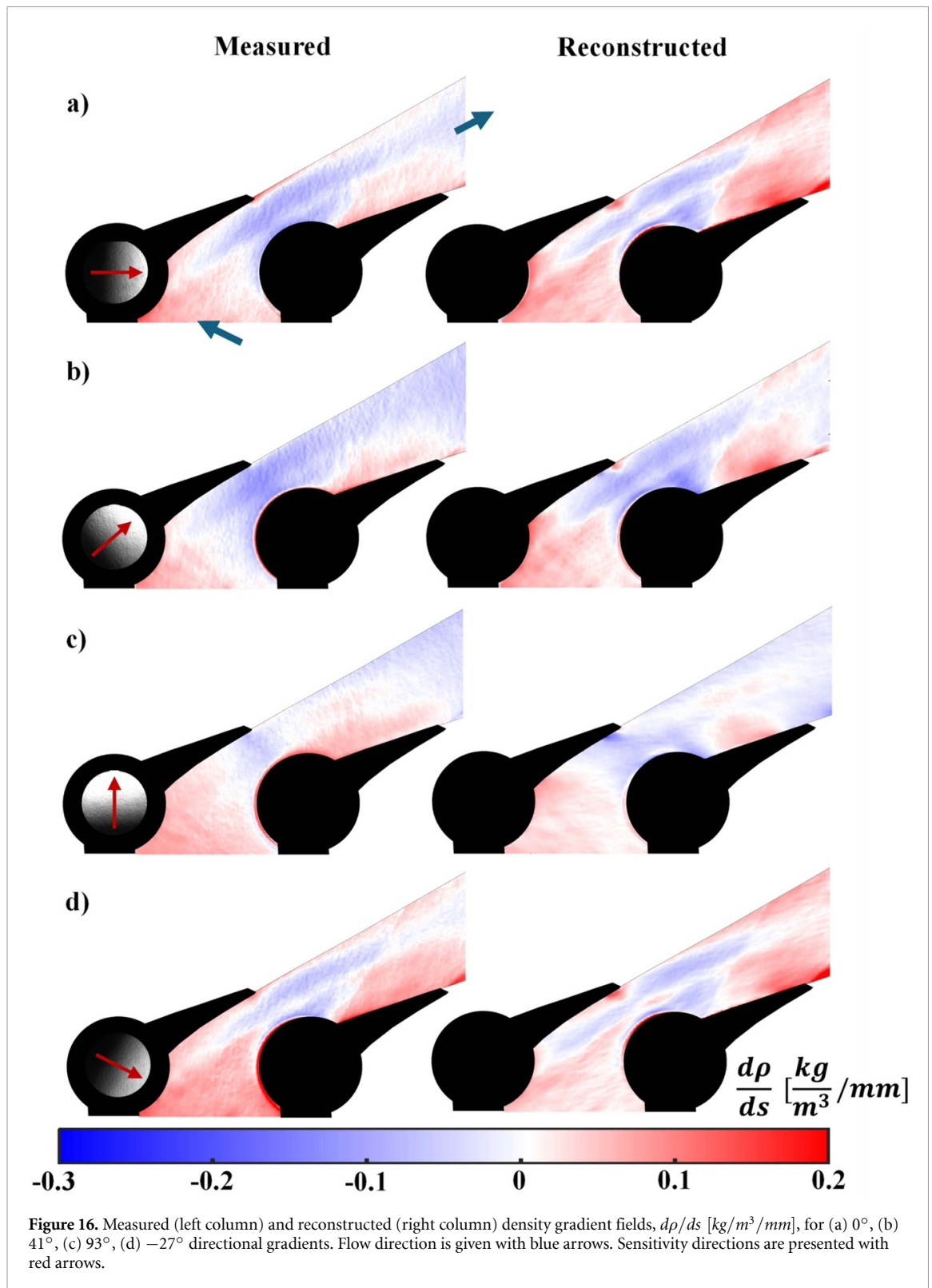
Orientation ablation was used to quantify the benefit of directional redundancy. The complete four-orientation set gives the lowest mean directional-gradient RMSE of 0.042, compared with 0.045–0.052 for the tested three-direction subsets and 0.053 for the near-orthogonal  $0^\circ/93^\circ$  pair. The nominal direction matrices remain well conditioned:  $\kappa_2(A) = 1.205$  for the complete set and 1.054 for the near-orthogonal pair. The smaller condition number of the two-direction pair is expected because it is already close to an ideal  $2 \times 2$  geometric configuration; the benefit of the additional directions is statistical redundancy rather than a lower condition number. Under an equal-variance directional-noise model, GRMS decreases from 1.001 for the  $0^\circ/93^\circ$  pair to 0.719 for the four-direction set, corresponding to approximately 30% lower RMS component-noise gain and 50% lower total estimator variance. Reduced-direction reconstructions also predict directions excluded from gradient estimation:  $0^\circ$ ,  $41^\circ$ , and  $-27^\circ$  predict the withheld  $93^\circ$  field with RMSE 0.040, while  $0^\circ$ ,  $93^\circ$ , and  $41^\circ$  predict the withheld  $-27^\circ$  field with RMSE 0.060; the two-direction cases predict excluded orientations with RMSE values of 0.049–0.062 (see table 1). These tests provide out-of-sample validation of the recovered gradient field.

## 6. Conclusion

This study presented a framework for reconstructing steady effective line-of-sight-averaged density fields in transonic flows from multi-directional calibrated monochrome Schlieren measurements. The algorithm was built on least-squares fusion of overdetermined gradient projections and a Poisson solver using FEMs with mixed boundary conditions. To reduce the impact of directional derivative inconsistencies, Dirichlet anchor points were relaxed using a uniform regularization factor, which softens boundary enforcement and prevents discontinuities.

Synthetic benchmarking demonstrated that the reconstruction algorithm maintains high accuracy and robustness against both noise and anchor point sparsity. The regularization parameter  $\lambda$  was identified as a critical tuning variable: under low noise conditions, enforcing the Dirichlet constraints rigidly yields the best reconstruction accuracy, whereas moderate to high noise levels favor intermediate values of in the range of 0.02–0.1, with the optimal value shifting toward smaller magnitudes as noise levels increase. This trend reflects a bias–variance trade-off: weaker anchor enforcement is preferred as the anchor data become less reliable, so that the gradient-derived field receives greater relative weight in the reconstruction.

The framework was applied to a transonic turbine cascade, combining four distinct gradient orientations with sparse anchor points derived from static pressure measurements on the blade surfaces. The reconstructed density field captured the principal flow structures through the blade passage: the acceleration of the flow manifested until the throat (indicated as a density decrease), followed by deceleration on the suction-side (indicated by a density rise), and ensuing wake velocity deficit (in this case, manifested as higher density streaks), and final mixing towards uniform density recovery at the exit conditions. This feature was consistently reproduced across all four sensitivity angles, as well as the reconstructed scalar field. The regenerated-gradient comparison demonstrates in-sample consistency with the



four measured projections, while reduced-direction reconstructions provide out-of-sample validation of the recovered gradient field; the absolute scalar-density accuracy is established by the synthetic ground-truth benchmarks.

The proposed methodology provides a versatile and scalable tool for density-field reconstruction in complex flow environments. The framework can be extended beyond turbine cascade applications to other high-speed flows as well as low-speed regimes involving temperature gradients. Reconstruction accuracy can be further improved by increasing the number of gradient measurement directions and enhancing the spatial coverage and precision of anchor points. The method's robustness to noise and

**Table 1.** Conditioning, noise amplification, and per-direction/mean spatial RMSE between measured and reconstructed density gradients for different subsets of knife-edge orientations.

| Used knife-edge angles | RMSE<br>(0°) | RMSE<br>(93°) | RMSE<br>(41°) | RMSE<br>(−27°) | Mean RMSE | $\kappa_2(A)$ |
|------------------------|--------------|---------------|---------------|----------------|-----------|---------------|
| 0°, 93°, 41°, −27°     | 0.042        | 0.035         | 0.051         | 0.040          | 0.042     | 1.205         |
| 0°, 41°, −27°          | 0.044        | 0.040         | 0.055         | 0.040          | 0.045     | 1.936         |
| 0°, 93°, 41°           | 0.052        | 0.042         | 0.053         | 0.060          | 0.052     | 1.362         |
| 41°, −27°              | 0.061        | 0.049         | 0.055         | 0.044          | 0.052     | 1.483         |
| 0°, 93°                | 0.054        | 0.046         | 0.052         | 0.062          | 0.053     | 1.054         |

sparse boundary data, combined with its generalizable formulation, positions it as a practical diagnostic tool for quantitative flow visualization across a broad range of aerodynamic and aerothermal experiments.

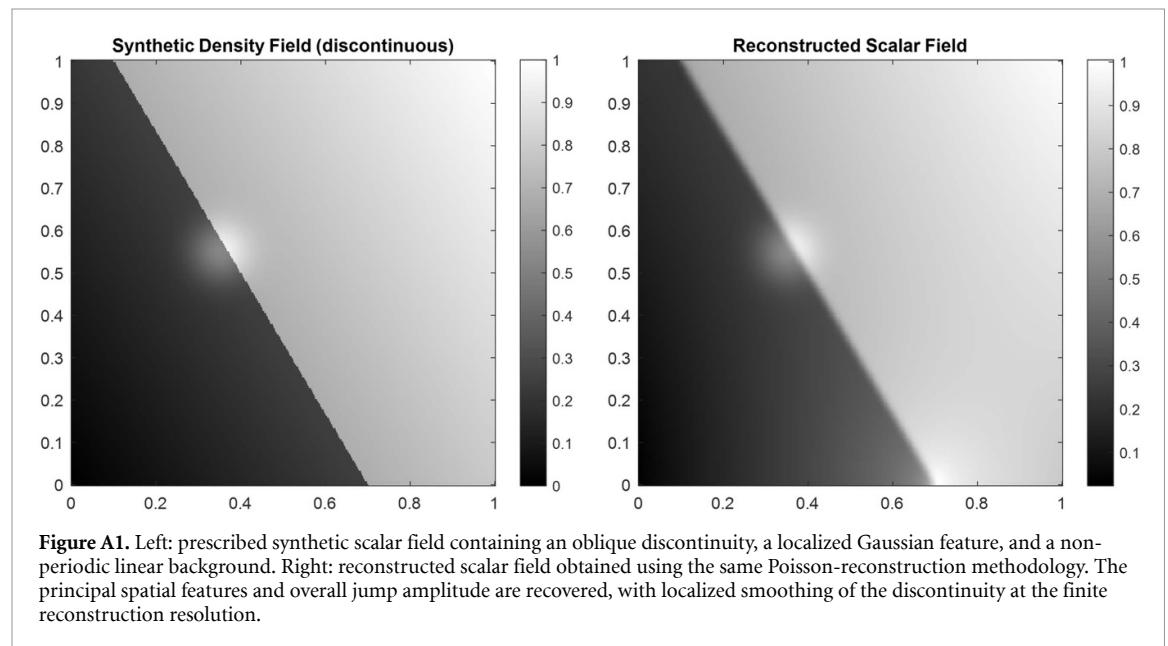
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## Data availability statement

The data cannot be made publicly available upon publication because they are not available in a format that is sufficiently accessible or reusable by other researchers. The data that support the findings of this study are available upon reasonable request from the authors.

## Appendix



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